Mathematics 272 Lecture 2 Notes

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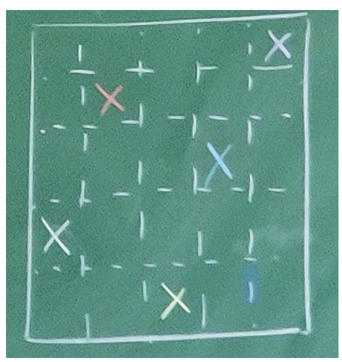
1 Convergence of Permutations

1.1 Subpermutations and permutation density

Definition 1.1. A **permutation of order** k is a bijection $[k] \to [k]$, where $[k]$ denotes the set $\{1, \dots, k\}$.

A permutation gives rise to two linear orders: the order of the actual numbers and the order we write the numbers in. We can express these two in a grid.

Example 1.1. The permutation $\boxed{2\ 4\ 1\ 3\ 5}$ can be expressed with the grid



Definition 1.2. A **subpermutation** (or a **pattern**) can be constructed by taking a subset of the numbers in the permutation and keeping them in order.

Example 1.2. Picking the 2nd, 3rd, and 5th numbers of the permutation $\boxed{2\ 4\ 1\ 3\ 5}$ gives the permutation $\boxed{2\ 1\ 3}$.

Example 1.3. Picking the 1st, 2nd, and 4th numbers of the permutation $\boxed{2\ 4\ 1\ 3\ 5}$ gives the permutation $\boxed{1\ 3\ 2}$.

Denote $|\pi|$ as the order of the permutation π.

Definition 1.3. The **density** of a permutation π in a permutation σ is

$$d(\pi, \rho) = \text{prob } |\pi| \text{ randomly chosen elements of } \sigma \text{ yield } \pi \text{ as a subpermutation.}$$

We define $d(\pi, \sigma) = 0$ when $|\pi| > |\sigma|$.

Example 1.4.

$$d(\ 1\ \ 2\ ,\ 2\ \ 4\ \ 1\ \ 3\ \ 5\ ) = \frac{7}{10}.$$

Example 1.5.

$$d(\ 1\ \ 2\ \ 3\ ,\ 2\ \ 4\ \ 1\ \ 3\ \ 5\ ) = \frac{3}{10}.$$

Definition 1.4. A sequence $(\pi_n)_{n\in\mathbb{N}}$ of permutations is **convergent** if $|\pi_n| \to \infty$ and for every permutation σ, $d(\sigma, \pi_n)$ converges.

1.2 Reducing convergence of permutations to 4-point permutation embeddings

We would like to prove the following theorem, but we will prove 2 weaker versions before proving the full theorem.

Theorem 1.1. *If $(\pi_n)_{n\in\mathbb{N}}$ is a sequence of permutations with $|\pi_n| \to \infty$ such that for all 4-point permutation σ, $d(\sigma, \pi_n) \to \frac{1}{24}$, then $(\pi_n)_{n\in\mathbb{N}}$ is convergent, and for every permutation τ, $d(\tau, \pi_n) \to \frac{1}{|\tau|!}$.*

Can the 4 be replaced with another number? This number is "optimal" in the sense that it cannot be replaced with 3 and have the theorem still hold true. Suppose we had all the densities of 3-point permutations in a permutation π:

$$d(\ 1\ \ 2\ \ 3\ ,\pi), \qquad d(\ 1\ \ 3\ \ 2\ ,\pi), \qquad d(\ 2\ \ 1\ \ 3\ ,\pi),$$

$$d(\ 2\ \ 3\ \ 1\ ,\pi), \qquad d(\ 3\ \ 1\ \ 2\ ,\pi), \qquad d(\ 3\ \ 2\ \ 1\ ,\pi).$$

Can we determine the value of $d(\ 1\ \ 2\ ,\pi)$?

Yes. We can do this by picking 3 elements and then picking 2 elements from those 3:

$$\begin{aligned} d(\ 1\ \ 2\ ,\pi) = \frac{3}{3}d(\ 1\ \ 2\ \ 3\ ,\pi) + \frac{2}{3}d(\ 1\ \ 3\ \ 2\ ,\pi) + \frac{2}{3}d(\ 2\ \ 1\ \ 3\ ,\pi) \\ + \frac{1}{3}d(\ 2\ \ 3\ \ 1\ ,\pi) + \frac{1}{3}d(\ 3\ \ 1\ \ 2\ ,\pi). \end{aligned}$$

This generalizes to the situation where $|\sigma| \le k \le |\pi|$:

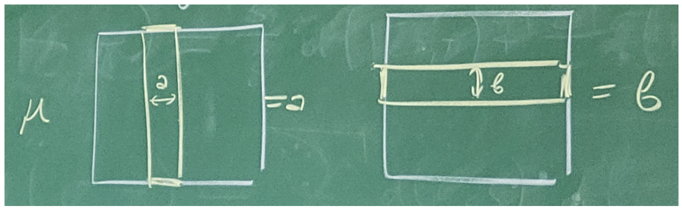
$$d(\sigma, \pi) = \sum_{\tau \in S_k} d(\sigma, \tau) d(\tau, \pi).$$

Remark 1.1. This result was proven for the number 5 by Hoeffding in 1948. He didn't use combinatorial limits to prove this theorem, however.

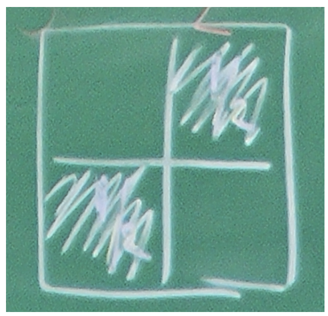
1.3 Permutons

We've mentioned convergence of permutations, but what is the limit object?

Definition 1.5. A **permuton** μ is a probability measure on Borel sets on $[0,1]^2$ that has uniform marginals.

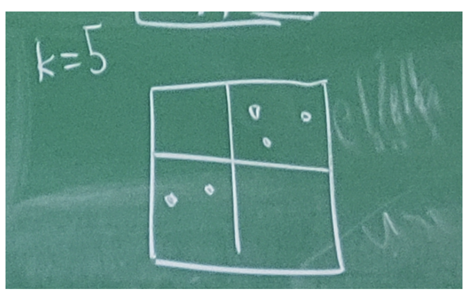


Example 1.6. Here is an example of a permuton which is not the uniform distribution on the square.



Definition 1.6. A **μ-random permutation of order** k is given by sampling k-points iid according to μ and interpreting it as a permutation in the reverse of the grid construction.

Example 1.7. The following sample with $k = 5$ gives the permutation $1 \quad 2 \quad 5 \quad 3 \quad 4$.



We define $d(\sigma, \mu)$ to be the probability that a μ-random permutation of order $|\sigma|$ is σ.

Proposition 1.1. *If $(\pi_n)_n$ is a convergent sequence of permutations, then there exists a unique permuton μ such that for every permutation σ, $\lim_{n\to\infty} d(\sigma, \pi_n) = d(\sigma, \mu)$.*

Here, μ is called the limit of the sequence $(\pi_n)_{n\in\mathbb{N}}$. We will later prove the following.

Proposition 1.2. *Let μ be a permuton, and let π_n be a μ-random permutation of order n. Then with probability one, $(\pi_n)_{n\in\mathbb{N}}$ is convergent, and μ is its limit.*

Proof of Proposition 1.1. Uniqueness is left as an exercise, but we will prove existence. Fix a convergent sequence $(\pi_n)_{n\in\mathbb{N}}$ of permutations, and for each n, look at the largest k such that $|\pi_n| \geq 2^{2k}$. Define $\pi'_n = \pi_n$ restricted to its first $|\pi_n| - (|\pi_n| \mod 2^k)$ elements. The point is to throw away a few elements and have 2^k divide $|\pi'_n|$. This does not lose the property of $|\pi'_n| \to \infty$, and for every permutation σ, we claim that

$$\lim_{n\to\infty} d(\sigma,\pi_n) = \lim_{n\to\infty} d(\sigma,\pi'_n).$$

This is because for every k, there exists an n_k such that for all $n \geq n_0$, 2^k divides $|\pi'_n|$. From that point on, $\frac{|\pi_n|-|\pi'_n|}{|\pi_n|} \leq 2^{-k}$. Also, because hitting one of the "removed" elements when sampling elements is small, we have

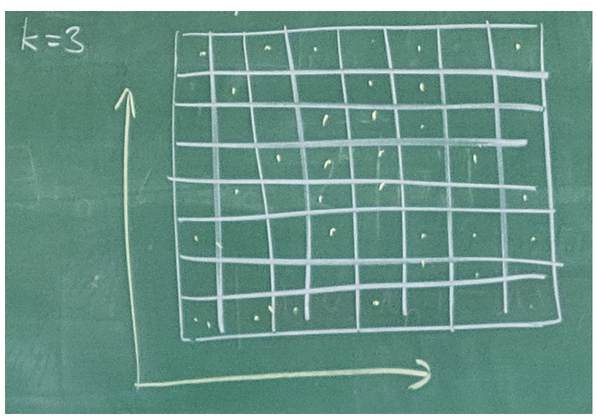
$$|d(\sigma,\pi_n) - d(\sigma,\pi'_n)| \leq |\sigma| \cdot 2^{-k} \cdot 2.$$

Thus, without loss of generality, we may assume that for every k, there exists an n_k such that for all $n \geq n_k$, 2^k divides $|\pi_n|$. We assume that $(\pi_n)_{n\in\mathbb{N}}$ has this property and drop the prime notation.

For every k, define a matrix $A_n^k \in [0,1]^{2^k\times 2^k}$ for every $n \geq n_k$ as

$$[A_n^k]_{i,j} := \frac{|\{x : \frac{j-1}{2^k}|\pi_n| < x \leq \frac{j}{2^k}|\pi_n| \text{ and } \frac{i-1}{2^k}|\pi_n| < x \leq \frac{i}{2^k}|\pi_n|\}|}{|\pi_n|}.$$

That is, we count the density of points landing in each box, indexing from left to right and bottom to top as in the following picture:



These matrices actually converge coordinate-wise. But rather than proving that, we will find a subsequence $(\pi_{m_i})_{i\in\mathbb{N}}$ such that for every k, $A_{m_i}^k$ converges. Let A^k be the limit matrix. Then A^1 is a $[0,1]^{2\times 2}$ matrix, $A^2 \in [0,1]^{4\times 4}$, and $A^1_{1,1} = A^2_{1,1} + A^2_{1,2} + A^2_{2,1} + A^2_{2,2}$. That is, we are subdividing each box into 4 smaller boxes and splitting up the density as we do so. Now, we would like to use Carathéodory's extension theorem:

Theorem 1.2. *Let $\mathscr{A}$ be an algebra of subsets of a set, i.e. it is closed under complements, finite unions, finite intersections, and $\varnothing \in \mathscr{A}$. Suppose we have a mapping $\mu_0 : \mathscr{A} \to \mathbb{R}_0^+$ such that if $(A_i)_{i\in\mathbb{N}}$ is an infinite sequence of disjoint subsets contained in $\mathscr{A}$ such that $\bigcup_{i=1}^\infty A_i \in \mathscr{A}$, then*

$$\mu_0\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu_0(A_i).$$

Then there exists a unique measure extending μ_0 to the σ-algebra generated by $\mathscr{A}$.

In our context, the parent set is $[0,1]\times[0,1]$, and

$$\mathscr{A} = \{\text{finite unions of half-open dyadic squares of the same order}\},$$

i.e. sets like $[\frac{x-1}{2^k}, \frac{x}{2^k}) \times [\frac{y-1}{2^k}, \frac{y}{2^k})$. Then we can define μ_0 to be the sum of the corresponding entries in A^k. Applying the theorem gives us the desired measure, and we will prove the permuton properties next time. $\square$